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THE INVARIANT SUBSPACE STRUCTURE OF NONSELFADJOINT CROSSED PRODUCTS

BY

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ABSTRACT. Let $\mathfrak{L}$ be the von Neumann algebra crossed product determined by a finite von Neumann algebra M and a trace preserving *-automorphism α of M. We study the invariant subspace structure of the subalgebra $\mathfrak{L}_+$ of $\mathfrak{L}$ consisting of those operators whose spectrum with respect to the dual automorphism group on $\mathfrak{L}$ is nonnegative. We investigate the conditions for two invariant subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (with Q_1, Q_2 the corresponding orthogonal projections) to satisfy $Q_1 = R_v Q_2 R_v^*$ for some partial isometry R_v in $\mathfrak{L}'$. We use this analysis to find the general form of a σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $\mathfrak{L}_+$.

1. Introduction. This paper extends some of the results of [**2, 3, 4** and **8**]. We are interested in the invariant subspace structure of certain subalgebras of von Neumann algebras constructed as crossed products of finite von Neumann algebras by trace preserving automorphisms. These subalgebras are called nonselfadjoint crossed products.

The setting here is the following. Let M be a von Neumann algebra with a faithful normal finite and normalized trace φ, and let α be a *-automorphism of M such that $\varphi \circ \alpha = \varphi$. Form the Hilbert space $L^2 = l^2(\mathbf{Z}) \otimes L^2(M, \varphi)$ and consider the operators L_x, $x \in M$, and L_δ defined on L^2 by the formulae $L_x = I \otimes x$ and $L_\delta = S \otimes u$, where S is the usual bilateral shift on $l^2(\mathbf{Z})$, and u is the unitary operator on $L^2(M, \varphi)$ that implements α. The *von Neumann algebra crossed product* determined by M and α is defined to be the von Neumann algebra $\mathfrak{L}$ generated by $\{L_x : x \in M\}$ $(= \mathfrak{L}(M))$ and L_δ, while the *nonselfadjoint crossed product* is the σ-weakly closed subalgebra $\mathfrak{L}_+$ generated by $\mathfrak{L}(M)$ and the positive powers of L_δ. Let H^2 be the subspace $l^2(\mathbf{Z}_+) \otimes L^2(M, \varphi)$ of L^2. We say that a subspace $\mathfrak{M}$ of L^2 is *invariant* if $\mathfrak{L}_+ \mathfrak{M} \subseteq \mathfrak{M}$. It is *pure* if $\bigcap_{n \geq 0} L_\delta^n \mathfrak{M} = \{0\}$. If every pure invariant subspace $\mathfrak{M}$ is of the form $\mathfrak{M} = R_v H^2$, where R_v is a partial isometry in the commutant of $\mathfrak{L}$, we shall say that the BLH theorem (i.e., the Beurling-Lax-Halmos theorem) is valid. (The BLH theorem is usually regarded as describing the invariant subspaces of the unilateral shift.)

In [**4**] it is shown that the following three conditions are equivalent:

(1) α acts trivially on the center of M.

(2) The BLH theorem is valid.

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(3) Each σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $\mathfrak{L}_+$ is of the form $L_e\mathfrak{L} \oplus (1 - L_e)\mathfrak{L}_+$, where e is a projection in the center of M and $\alpha(e) = e$.

In general, the BLH theorem is not valid. What we can do is study the conditions for two invariant subspaces $\mathfrak{M}_1$, $\mathfrak{M}_2$ to satisfy $\mathfrak{M}_1 = R_v\mathfrak{M}_2$ for some partial isometry $R_v \in \mathfrak{L}'$. In [**2**], M. McAsey used the notion of multiplicity function (associated with a pure invariant subspace) in the case where M is commutative and found that, in this case, two pure invariant subspaces $\mathfrak{M}_1$, $\mathfrak{M}_2$, with multiplicity functions m_1, m_2 respectively, satisfy: $m_1 \leq m_2$ if and only if $Q_1 = R_vQ_2R_v^*$ (where Q_i is the projection onto $\mathfrak{M}_i$) for some partial isometry $R_v \in \mathfrak{L}'$. The multiplicity functions of pure invariant subspaces were further studied in [**8**].

Our objective here is to extend this notion to the noncommutative case using an operator valued trace on $\mathfrak{L}(M)'$ (see §2). Doing this, we manage to associate, with each pure invariant subspace $\mathfrak{M}$, a function $\phi(P(\mathfrak{M}))$ on the maximal ideal space X of the center of M, such that $\phi(P(\mathfrak{M}_1)) \leq \phi(P(\mathfrak{M}_2))$ if and only if $Q_1 = R_vQ_2R_v^*$ for some partial isometry $R_v \in \mathfrak{L}'$ (so that $\mathfrak{M}_1 = R_v\mathfrak{M}_2$).

The validity of the BLH theorem is related to the maximality of $\phi(P(H^2))$ among the functions that can be obtained as $\phi(P(\mathfrak{M}))$ for some pure invariant subspace $\mathfrak{M}$. This set of functions is completely described in Theorem 3.7, which is the main result of §3.

In §4 we find the general form of a σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $\mathfrak{L}_+$ (see Theorem 4.5).

We also show (in Proposition 3.10 and Corollary 4.6) that our results really extend results of [**3** and **4**].

We tried to keep our definitions and notations as close as possible to those of [**3**]. We also want to note that throughout the paper, "a projection" always means "an orthogonal projection" and "a subspace" means "a closed subspace."

2. Preliminaries and the definition of ϕ. Let M be a finite von Neumann algebra with a faithful and normal finite trace φ. We will assume that M is in standard form and identify it with the von Neumann algebra of left multiplictions on $L^2(M, \varphi)$ (see [7]). The algebra M' is its commutant on $L^2(M, \varphi)$. Since M has a generating and separating vector, M' is also finite. We will write Z for $M \cap M'$ and identify it with $L^\infty(X, \nu)$ for some locally compact Hausdorff space X with a probability measure ν such that

$$\int_X f\,d\nu = \varphi(f), \qquad f \in L^\infty(X, \nu).$$

We fix once and for all a normal, *-automorphism α of M which preserves φ; i.e., $\varphi \circ \alpha = \varphi$. The following proposition appears in [**3**].

PROPOSITION 2.1. *Let* $L_0^2 = \{f\colon \mathbf{Z} \to M;\ f(n) = 0$ *for all but finitely many* $n\}$. *Then with respect to pointwise addition and scalar multiplication and the operations defined by equations* (1)–(3), L_0^2 *is a Hilbert algebra with identity* ψ *defined by* $\psi(0) = I_M$, *and* $\psi(n) = 0$, $n \neq 0$.

(1) $$(f*g)(n)=\sum_{k\in Z} f(k)\alpha^k(g(n-k)),$$

(2) $$(f^*)(n)=[\alpha^n(f(-n))]^*,$$

(3) $$\langle f,g\rangle=\sum_{k\in Z}(f(k),g(k))_{L^2(M,\varphi)}.$$

Note that the Hilbert space completion L^2 of L_0^2 is $\{f\colon \mathbf{Z}\to L^2(M,\varphi);\ \sum_{n\in\mathbf{Z}}\|f(n)\|^2_{L^2(M,\varphi)}<\infty\}$.

For f in L_0^2, we define operators L_f and R_f on L^2 by $L_f g=f*g$ and $R_f g=g*f$, $g\in L^2$. Both L_f and R_f are well-defined, bounded operators, and we set $\mathfrak{L}=\{L_f\colon f\in L_0^2\}''$, $\mathfrak{R}=\{R_f\colon f\in L_0^2\}''$. Also, we define L^∞ to be the achieved Hilbert algebra of all bounded elements in L^2. For such an f, we write L_f and R_f for the operators it determines. It is known that the map $f\to L_f$ [resp. $f\to R_f$] is a *-isomorphism [resp. *-anti-isomorphism] from L^∞ onto $\mathfrak{L}$ [resp. $\mathfrak{R}$]. Moreover, $\mathfrak{L}$ and $\mathfrak{R}$ are finite von Neumann algebras with $\mathfrak{R}'=\mathfrak{L}$. We call L^∞ the *selfadjoint* or *von Neumann algebra crossed product* determined by M, φ, and α and refer to $\mathfrak{L}$ and $\mathfrak{R}$ as the left and right regular representations of it.

The original algebra M is identified with the subalgebra $\{x\psi\colon x\in M\}$ of L^∞, and we write L_x (and R_x) for $L_{x\psi}$ (and $R_{x\psi}$). We have, for $f\in L^2$,

$$(L_x f)(n)=xf(n)\quad\text{and}\quad (R_x f)(n)=f(n)\alpha^n(x).$$

We write $\mathfrak{L}(M)=\{L_x\colon x\in M\}$ and $\mathfrak{R}(M)=\{R_x\colon x\in M\}$.

If we let δ be defined by $\delta(n)=0$ if $n\neq 1$, $\delta(1)=I_M$, then it is easy to check that $\mathfrak{L}$ is the von Neumann algebra generated by $\mathfrak{L}(M)$ and L_δ and, similarly, $\mathfrak{R}$ is generated by $\mathfrak{R}(M)$ and R_δ.

The automorphism group $\{\beta_t\}_{t\in\mathbf{R}}$ of $\mathfrak{L}$ dual to α in the sense of Takesaki [**9**] is implemented by the unitary representation of $\mathbf{R}$, $\{W_t\}_{t\in\mathbf{R}}$, defined by

$$(W_t f)(n)=e^{2\pi int}f(n),\qquad f\in L^2;$$

that is, $\beta_t(L_f)=W_tL_fW_t^*$. Similarly, $\beta_t(R_f)=W_tR_fW_t^*$. It is easy to see that $\beta_t(L_f)=L_{W_tf}$ for f in L^∞ and similarly for R_f. One can check that the spectral resolution of $\{W_t\}_{t\in\mathbf{R}}$ is given by

$$W_t=\sum_{n=-\infty}^{\infty}e^{2\pi int}E_n,$$

where E_n is the projection on L^2 defined by

$$(E_nf)(k)=\begin{cases} f(n), & k=n,\\ 0, & k\neq n.\end{cases}$$

The restriction of E_n to L^∞ will be denoted by ε_n, and we shall write $\varepsilon_n(L_f)=L_{\varepsilon_n(f)}$ and $\varepsilon_n(R_f)=R_{\varepsilon_n(f)}$. We have

$$\varepsilon_n=\int_0^1 e^{-2\pi int}\beta_t\,dt,$$

where the integral converges in the σ-weak topology when applied to operators.

We define $H^2 = \{f \in L^2: f(n) = 0, n < 0\}$, and we let H^∞ be $L^\infty \cap H^2$. We refer to H^∞ as the nonselfadjoint crossed product determined by M and α. Also, we set $\mathcal{L}_+ = \{L_f: f \in H^\infty\}$ and $\mathcal{R}_+ = \{R_f: f \in H^\infty\}$.

The algebra $\mathcal{L}_+$ (resp. $\mathcal{R}_+$) is the σ-weakly closed algebra generated by L_δ and $\mathcal{L}(M)$ (resp. R_δ and $\mathcal{R}(\mathfrak{M})$) (see [**3**, Theorem 2.2]).

In studying the invariant subspaces of $\mathcal{L}_+$ we will need the following definitions.

We shall say that a subspace $\mathfrak{M}$ of L^2 is: *left-invariant*, if $\mathcal{L}_+ \mathfrak{M} \subseteq \mathfrak{M}$; *left-reducing*, if $\mathcal{L}\mathfrak{M} \subseteq \mathfrak{M}$; *left-pure*, if $\mathfrak{M}$ contains no left-reducing subspace; and *left-full* if the smallest left-reducing subspace containing $\mathfrak{M}$ is L^2 (throughout the paper, a subspace means a closed subspace). The right-hand versions are defined similarly, and a subspace that is both left- and right-invariant will be called *two-sided invariant*. In order to shorten the writing, whenever we refer to a subspace as being invariant, reducing, pure or full, we mean that it is left-invariant, left-reducing, etc. The following result is Proposition 3.1 in [**3**].

PROPOSITION 2.2. *Let $\mathfrak{M}$ be an invariant subspace in L^2. Then*
(1) *$\mathfrak{M}$ reduces $\mathcal{L}(M)$;*
(2) *$\mathfrak{M}$ reduces $\mathcal{L}$ if and only if $\mathfrak{M}$ reduces L_δ;*
(3) *$\mathfrak{M}$ is pure if and only if $\bigwedge_{n \geq 0} L_\delta^n \mathfrak{M} = \{0\}$; and*
(4) *$\mathfrak{M}$ is full if and only if $\bigvee_{n \leq 0} L_\delta^n \mathfrak{M} = L^2$.*

If $\mathfrak{M}$ is an invariant subspace, then the subspace $\mathfrak{N} = \mathfrak{M} \ominus L_\delta \mathfrak{M}$ is a *wandering subspace*; i.e., $L_\delta^n \mathfrak{N}$ and $L_\delta^m \mathfrak{N}$ are orthogonal when $n \neq m$. We can write $\mathfrak{M} = \sum_{n=0}^\infty \oplus L_\delta^n \mathfrak{N} \oplus \mathfrak{M}_\infty$, where $\mathfrak{M}_\infty = \bigwedge_{n \geq 0} L_\delta^n \mathfrak{M}$. Clearly $\mathfrak{M}_\infty$ is a reducing subspace and $\sum_{n=0}^\infty \oplus L_\delta^n \mathfrak{N}$ is a pure invariant subspace. Since each reducing subspace is of the form $R_e L^2$ for a suitable projection e in L^∞ (see [**5**]), the analysis of the invariant subspace structure of $\mathcal{L}_+$ may be reduced to the analysis of the pure invariant subspaces. For such a subspace $\mathfrak{M}$, denote by $P(\mathfrak{M})$ the orthogonal projection onto $\mathfrak{M} \ominus L_\delta \mathfrak{M}$. A projection on a wandering subspace will be called a *wandering projection*. Thus, with every pure invariant subspace $\mathfrak{M}$ we have associated a wandering projection $P(\mathfrak{M})$ such that the range of the projection $\sum_{n=0}^\infty L_\delta^n P(\mathfrak{M}) L_\delta^{-n}$ is $\mathfrak{M}$. Also note that, since $\mathfrak{M}$ is invariant, it reduces $\mathcal{L}(M)$ and, therefore, $P(\mathfrak{M})$ lies in $\mathcal{L}(M)'$. Conversely, if P, in $\mathcal{L}(M)'$, is a wandering projection, then the range of $\sum_{n=0}^\infty L_\delta^n P L_\delta^{-n}$, denoted $\mathfrak{M}(P)$, is a pure invariant subspace.

It is shown in [**3**, Theorem 3.2] that when $\mathfrak{M}_i$, $i = 1, 2$, is a pure invariant subspace of L^2, and Q_i is the orthogonal projection onto $\mathfrak{M}_i$, then $P(\mathfrak{M}_2) \precsim P(\mathfrak{M}_1)$ (in $\mathcal{L}(M)'$) if and only if there is a partial isometry R_v in $\mathcal{R}$ such that $Q_2 = R_v Q_1 R_v^*$ (in this event $\mathfrak{M}_2 = R_v \mathfrak{M}_1$). One can also check, following the proof of this result, that if $P(\mathfrak{M}_1) \sim P(\mathfrak{M}_2)$, then we can have $R_v R_v^* \geq Q_2$ and $R_v^* R_v \geq Q_1$.

Therefore, in order to analyze the invariant subspace structure we will study the relation "$\precsim$" among the wandering projections of $\mathcal{L}(M)'$. We will do this using the $\mathcal{L}(Z)$-trace on $\mathcal{L}(M)'$ (where $\mathcal{L}(Z) = \{L_c: c \in Z\} = \mathcal{L}(M) \cap \mathcal{L}(M)'$). Let

$$\mathcal{P}_1 = \{P \in \mathcal{L}(M)': P \text{ is a wandering projection}\}.$$

We will define an $\mathcal{L}(Z)$-trace following [**1**, Chapter III, §4]. The algebra $\mathcal{L}(Z)$ is *-isomorphic to the algebra $L^\infty(X, \nu)$ (since Z is *-isomorphic to $\mathcal{L}(Z)$). Define $\mathfrak{T}$ to be the set of nonnegative measurable functions, finite or not, on X (we identity two functions in $\mathfrak{Z}$ if they are different only on a set of measure zero). In some cases we will write $\mathfrak{L}(\mathfrak{Z})$ instead of $\mathfrak{Z}$. This should not cause any confusion since Z can be identified with $\mathfrak{L}(Z)$.

DEFINITION. An $\mathfrak{L}(Z)$-*trace* on $\mathfrak{L}(M)'_+$ (the positive cone of $\mathfrak{L}(M)'$) is a map ϕ, defined on $\mathfrak{L}(M)'_+$, with values in $\mathfrak{Z}$, satisfying:

(1) If $T, S \in \mathfrak{L}(M)'_+$, then $\phi(S+T) = \phi(S) + \phi(T)$.

(2) If $S \in \mathfrak{L}(M)'_+$ and $T \in \mathfrak{L}(Z)_+$, then $\phi(TS) = T\phi(S)$.

(3) If $S \in \mathfrak{L}(M)'_+$ and $U \in \mathfrak{L}(M)'$ is a unitary operator, then $\phi(USU^*) = \phi(S)$.

We say that ϕ is *faithful* if whenever S lies in $\mathfrak{L}(M)'_+$ and $\phi(S) = 0$, then $S = 0$.

We say that ϕ is *finite* if $\phi(\mathfrak{L}(M)'_+) \subseteq \mathfrak{L}(Z)_+$ and *semifinite* if, for every $S \neq 0$ in $\mathfrak{L}(M)'_+$, there is an operator $T \in \mathfrak{L}(M)'_+$, $T \neq 0$, such that $T \leqslant S$ and $\phi(T) \in \mathfrak{L}(Z)_+$.

We say that ϕ is *normal* if, for each increasing net $\{S_\alpha\}$ in $\mathfrak{L}(M)'_+$, $\sup_\alpha \phi(S_\alpha) = \phi(\sup_\alpha S_\alpha)$.

We note that when we write just "trace", and not "$\mathfrak{L}(Z)$-trace", we always mean a numerical trace.

LEMMA 2.3. *The algebra $E_0\mathfrak{L}(M)'E_0$ is unitarily isomorphic to M'. In particular, E_0 is a finite projection in $\mathfrak{L}(M)'$.*

PROOF. The operator U: $E_0(L^2) \to L^2(M, \varphi)$ defined by $Uf = f(0)$ is an isometry from the range of E_0 onto $L^2(M, \varphi)$. It is easy to check that $U^*MU = \mathfrak{L}(M)E_0$. Thus $E_0\mathfrak{L}(M)'E_0 = (\mathfrak{L}(M)E_0)' = (U^*MU)' = U^*M'U$. $\square$

Since $\mathfrak{L}(M)'$ is a semifinite algebra, there is a semifinite normal faithful trace τ defined on $\mathfrak{L}(M)'_+$. Since φ, restricted to Z_+ $(= L^\infty(X, \nu)_+)$ is a semifinite normal faithful trace (on Z_+), there is a semifinite normal faithful trace ω defined on $\mathfrak{Z}$ that extends φ (see [**1**, pp. 244–245]). In fact, $\omega(S) = \int_X S\,d\nu$, $S \in \mathfrak{Z}$. Using [**1**, Proposition 3, p. 247], we can find a unique normal $\mathfrak{L}(Z)$-trace ϕ_0 such that $\tau = \omega \circ \phi_0$. Since τ is faithful, so is ϕ_0. Also, τ is semifinite and ω is faithful; hence ϕ_0 is semifinite (see [**1**, Proposition 2, p. 246]).

Since E_0 is a finite projection in $\mathfrak{L}(M)'$, we have $\tau(E_0) < \infty$; hence, $\int_X \phi_0(E_0)\,d\nu < \infty$ and $\phi_0(E_0) < \infty$, a.e. If $\phi_0(E_0)N = 0$ for some nonzero projection N in $\mathfrak{L}(Z)$, then $\phi_0(E_0N) = 0$, which implies, by faithfulness of ϕ_0, that $E_0N = 0$. This is impossible, since if $N = L_c$, $c \in Z$, $c \neq 0$, then $(E_0N\psi)(0) = c \neq 0$ (where $\psi(0) = I_M$, $\psi(n) = 0$ for $n \neq 0$). Thus, $0 < \phi_0(E_0) < \infty$, a.e.

This allows us to define another $\mathfrak{L}(Z)$-trace by

$$\phi(T) = \phi_0(T)\phi_0(E_0)^{-1}, \qquad T \in \mathfrak{L}(M)'_+ \quad .$$

It follows that ϕ is the unique, faithful, normal, semifinite $\mathfrak{L}(Z)$-trace that maps E_0 into I (for uniqueness see [**1**, Theorem 2, p. 248]).

The $\mathfrak{L}(Z)$-trace ϕ induces a map ρ, from $(E_0\mathfrak{L}(M)'E_0)_+$ into $(\mathfrak{L}(Z)E_0)_+$, by

$$\rho(T) = E_0\phi(T), \qquad T \in (E_0\mathfrak{L}(M)E_0)_+ .$$

The map ρ is actually a finite, faithful, normal, center-valued trace on $(E_0\mathfrak{L}(M)'E_0)_+$. To check that, for each $c \in (\mathfrak{L}(Z)E_0)_+$ (where $\mathfrak{L}(Z)E_0$ is the center of $E_0\mathfrak{L}(M)'E_0)$, $\rho(c)=c$, write $c=c_1E_0$ where $c_1\in\mathfrak{L}(Z)$. Then

$$\rho(c)=E_0\phi(c)=E_0\phi(c_1E_0)=c_1E_0\phi(E_0)=c_1E_0=c.$$

All the above-mentioned properties of ρ follow from the corresponding properties of ϕ.

Let $\mathcal{C}$ denote the set $\{g\in L^\infty(X,\nu);\ L_gE_0=\rho(P),\ P$ is a projection in $E_0\mathfrak{L}(M)'E_0\}$. Since $L_gE_0=L_fE_0$ implies $g=f$ $(g,f\in L^\infty(X,\nu))$, we have $\mathcal{C}=\{g\in L^\infty(X,\nu);\ L_g=\phi(P),\ P$ is a projection in $E_0\mathfrak{L}(M)'E_0\}$. Identifying $E_0\mathfrak{L}(M)'E_0$ with M', and $\mathfrak{L}(Z)$ with Z, we can say that $\mathcal{C}$ is the image of the dimension function on the projections of M'.

LEMMA 2.4. *Let Q be a projection in $E_0\mathfrak{L}(M)'E_0$ with $\phi(Q)=L_f$, $f\in Z$. Then $L_\delta QL_\delta^*$ lies in $\mathfrak{L}(M)'$ and $\phi(L_\delta QL_\delta^*)=L_{\alpha(f)}$.*

PROOF. Let ρ be the unique, faithful, normal, center-valued trace defined, as above, by $\rho(T)=E_0\phi(T)$. For $T\in E_0\mathfrak{L}(M)'E_0$, $R_\delta^*L_\delta TL_\delta^*R_\delta$ also lies in $E_0\mathfrak{L}(M)'E_0$ (since $R_\delta^*L_\delta E_0L_\delta^*R_\delta=R_\delta^*E_1R_\delta=E_0$), and we can define $\rho_1(T)=\alpha^{-1}(\rho(R_\delta^*L_\delta TL_\delta^*R_\delta))$, $T\in(E_0\mathfrak{L}(M)'E_0)_+$ (where $\alpha^{-1}(L_fE_0)$ is defined to be $L_{\alpha^{-1}(f)}E_0$). Then, for each $L_fE_0\in(\mathfrak{L}(Z)E_0)_+$,

$$\begin{aligned}\rho_1(L_fE_0)&=\alpha^{-1}\big(\rho\big(R_\delta^*L_\delta L_fE_0L_\delta^*R_\delta\big)\big)=\alpha^{-1}\big(\rho\big(R_\delta^*L_{\alpha(f)}R_\delta R_\delta^*E_1R_\delta\big)\big)\\&=\alpha^{-1}\big(\rho\big(L_{\alpha(f)}E_0\big)\big)=\alpha^{-1}\big(L_{\alpha(f)}E_0\big)=L_fE_0.\end{aligned}$$

Hence ρ_1 is also a center-valued trace and, thus, $\rho_1=\rho$, and this completes the proof. □

Suppose E, F are projections in $\mathfrak{L}(M)'$. From the trace properties of ϕ it follows that:

(1) $E\sim F$ implies $\phi(E)=\phi(F)$.

(2) $E\precsim F$ implies $\phi(E)\leqslant\phi(F)$.

(3) If $E\precsim F$ and $E\nsim F$, then $\phi(E)<\phi(F)$, providing $\phi(E)<\infty$.

(4) If $\phi(E)\leqslant\phi(F)<\infty$, then $E\precsim F$.

This is proved as follows: assume $E\not\precsim F$; then $QF\precsim QE$ and $QF\nsim QE$ for some projection Q in $\mathfrak{L}(Z)$; hence, $Q\phi(F)=\phi(QF)<\phi(QE)=Q\phi(E)$, which contradicts our hypothesis.

(5) If $\phi(E)=\phi(F)<\infty$, then $E\sim F$ (follows immediately from (4)).

COROLLARY 2.5. *Suppose $\mathfrak{M}_i$, $i=1,2$, is a pure invariant subspace such that $\phi(P(\mathfrak{M}_i))<\infty$. Then $\phi(P(\mathfrak{M}_2))\leqslant\phi(P(\mathfrak{M}_1))$ if and only if there is a partial isometry R_v, in $\mathfrak{R}$, such that $Q_2=R_vQ_1R_v^*$ (where Q_i is the projection onto $\mathfrak{M}_i$). (We will later see that the condition $\phi(P(\mathfrak{M}_i))<\infty$ is redundant.)*

3. What is $\phi(\mathfrak{P}_1)$? Define $\mathfrak{F}_0=\{f\in\mathfrak{Z}:\ f=\sum_{k=0}^\infty(k+g_k)N_k+\infty\cdot N_\infty,\ N_k$ are projections in $L^\infty(X,\nu)$ with $N_kN_j=0$, $k\neq j$, and $N_\infty+\sum_{k=0}^\infty N_k=I$, $g_k\in\mathcal{C}\}$. Also, let C denote the set of projections N in Z satisfying $\alpha(N)=N$, and let $\mathfrak{F}_1$ be $\{f\in\mathfrak{F}_0:\ \varphi(fN)\leqslant\varphi(N)$ whenever $N\in C\}$. The main result of this section is that $\phi(\mathfrak{P}_1)=\mathfrak{L}(\mathfrak{F}_1)$ $(=\{L_f:\ f\in\mathfrak{F}_1\})$. This will completely analyze the invariant space structure with respect to the relation $Q_2=R_vQ_1R_v^*$ where $\mathfrak{M}_i=Q_i(L^2)$ (see Corollary 2.5).

Now, let $\mathcal{P}_0$ be the set of projections in $\mathfrak{L}(M)'$.

LEMMA 3.1. $\phi(\mathcal{P}_0) = \mathfrak{L}(\mathcal{F}_0)$.

PROOF. We will use the following notation: whenever N is a projection in $L^\infty(X, \nu)$, $\hat{N}$ will denote a subset of X with $N = \chi_{\hat{N}}$ (unique when we ignore sets of measure zero). Conversely, when $\hat{N}$ is a subset of X, N is the corresponding projection in $L^\infty(X, \nu)$.

Suppose P lies in $\mathcal{P}_0$ and denote by P_n the projection $\sum_{k=0}^{n} E_k$ (in $\mathcal{P}_0$), then $\phi(P_n) = (n+1)I$. Since $\phi(P) \in \mathfrak{L}(\mathfrak{Z})$, $\phi(P) = L_f, f \in \mathfrak{Z}$. Let $\hat{N}_k$ be the set $\{x \in X: k \leq f(x) < k+1\}$ (well defined up to a set of measure zero). We have

$$kN_k \leq N_k f < (k+1)N_k, \qquad \infty > k \geq 0,$$

and

$$kL_{N_k} = \phi\left(L_{N_k}P_{k-1}\right) \leq \phi\left(L_{N_k}P\right) < \phi\left(L_{N_k}P_k\right) = (k+1)L_{N_k}.$$

Fix $0 \leq k < \infty$. Then there are $F, E \in \mathcal{P}_0$ such that

$$L_{N_k}P_{k-1} \sim F \leq L_{N_k}P \sim E < L_{N_k}P_k$$

and

$$0 \leq L_{N_k}P - F \prec L_{N_k}P_k - L_{N_k}P_{k-1} = L_{N_k}E_k \sim L_{N_k}E_0.$$

Hence $L_{N_k}P - F \sim Q \leq L_{N_k}E_0$ for some $Q \in \mathcal{P}_0$, and

$$\phi\left(L_{N_k}P\right) - kN_k = \phi\left(L_{N_k}P - F\right) = \phi(Q) \in \mathfrak{L}(\mathcal{C}).$$

Hence,

$$\phi\left(L_{N_k}P\right) = kL_{N_k} + L_{N_k}L_{g_k}, \qquad g_k \in \mathcal{C}.$$

This holds for each $0 \leq k < \infty$. For $k = \infty$ we have $\phi(L_N P) = \infty \cdot L_{N_\infty}$ (where $\infty \cdot 0 = 0 \cdot \infty = 0$); and, thus,

$$\phi(P) = \sum_{k=0}^{\infty} \phi\left(L_{N_k}P\right) + \phi\left(L_{N_\infty}P\right) \in \mathfrak{L}(\mathcal{F}_0).$$

For the other direction, let f be in $\mathcal{F}_0$, and let g_k be in $\mathcal{C}$ such that

$$f = \sum_{k=0}^{\infty} (k + g_k)N_k + \infty \cdot N_\infty,$$

where N_k are projections in $L^\infty(X, \nu)$, $N_k N_j = 0$ when $k \neq j$, and $N_\infty + \sum_{k=0}^{\infty} N_k = I$. In order to show that L_f lies in $\phi(\mathcal{P}_0)$, it will suffice to produce, for each $0 \leq k \leq \infty$, a projection Q_k in $\mathcal{P}_0$, with

$$\phi(Q_k) = kI + L_{g_k}, \quad k < \infty, \qquad \text{and} \qquad \phi(Q_\infty) = \infty.$$

Indeed, if $Q_k \in \mathcal{P}_0$ are such projections, then

$$\phi\left(\sum_{k=0}^{\infty} Q_k L_{N_k} + Q_\infty L_{N_\infty}\right) = L_f.$$

For $0 \leq k < \infty$, there is a projection $F_k \in E_0 \mathfrak{L}(M)' E_0$ with $\phi(F_k) = L_{g_k}$ (since $g_k \in \mathcal{C}$); then $\phi(F_k + \sum_{n=0}^{k-1} E_n) = kI + L_{g_k}$, so we let Q_k be $F_k + \sum_{n=0}^{k-1} E_n$. For $k = \infty$, let Q be I, and we are done. $\square$

LEMMA 3.2. (1) *If* $f \in \mathcal{C}$, *then* $\alpha(f) \in \mathcal{C}$.

(2) *If* $f_1, f_2 \in \mathcal{C}$, *then* $f_1 \wedge f_2 \in \mathcal{C}$ *and* $f_i - f_1 \wedge f_2 \in \mathcal{C}$.

(3) *If* $f_1, f_2 \in \mathcal{C}$, *then the function*

$$f(x) = \begin{cases} f_1(x) + f_2(x), & f_1(x) + f_2(x) \leqslant 1, \\ f_1(x) + f_2(x) - 1, & \textit{otherwise}, \end{cases}$$

lies in $\mathcal{C}$.

PROOF. (1) Let L_f be $\phi(Q)$, Q a projection in $E_0\mathfrak{L}(M)'E_0$. Then $L_{\alpha(f)} = \phi(R_\delta^* L_\delta Q L_\delta^* R_\delta)$ (by Lemma 2.4) and $R_\delta^* L_\delta Q L_\delta^* R_\delta$ also lies in $E_0\mathfrak{L}(M)'E_0$.

(2) If $\phi(Q_i) = L_{f_i}$, $i = 1, 2$, then $f = f_1 \wedge f_2 = N_1 f_1 + N_2 f_2$ for some projections N_1, N_2 in $L^\infty(X, \nu)$ with $N_1 N_2 = 0$, $N_1 + N_2 = I$. Thus $L_f = \phi(L_{N_1}Q_1 + L_{N_2}Q_2) \in \mathfrak{L}(\mathcal{C})$. For the other part, $f_1 - f_1 \wedge f_2 = N_2(f_1 - f_2)$ and, since $\phi(L_{N_2}Q_1) = L_{N_2}L_{f_1} \geqslant L_{N_2}L_{f_2} = \phi(L_{N_2}Q_2)$, there is some projection $E \in \mathcal{P}_0$ such that $L_{N_2}Q_1 \sim E \leqslant L_{N_2}Q_1$. We have $\phi(L_{N_2}Q_1 - E) = L_{N_2}(L_{f_1} - L_{f_2})$; hence, $f_1 - f_1 \wedge f_2 \in \mathcal{C}$. Similarly, $f_2 - f_1 \wedge f_2 \in \mathcal{C}$.

(3) For some projections N_1, N_2 in $L^\infty(X, \nu)$ with $N_1 N_2 = 0$, $N_1 + N_2 = I$, we have $f = N_1(f_1 + f_2) + N_2(f_1 + f_2 - 1)$ and $\phi(Q_i) = L_{f_i}$, $i = 1, 2$, $Q_i \in \mathcal{P}_0$, $Q_i \leqslant E_0$. Since

$$\phi(Q_1 L_{N_1}) = L_{f_1}L_{N_1} \leqslant L_{N_1} - L_{f_2}L_{N_1} = L_{N_1} - \phi(Q_2 L_{N_1}) = \phi(E_0 L_{N_1} - Q_2 L_{N_1}),$$

there is some $E \in \mathcal{P}_0$ with $Q_1 L_{N_1} \sim E \leqslant E_0 L_{N_1} - Q_2 L_{N_1}$. Therefore, $L_{N_1}(L_{f_i} + L_{f_2}) = \phi(L_{N_1}(Q_2 + E))$, so $N_1(f_1 + f_2) \in \mathcal{C}$. Similarly, $N_2(f_1 + f_2 - 1) \in \mathcal{C}$; hence $f \in \mathcal{C}$. □

As a consequence of Lemma 3.2, we obtain

LEMMA 3.3. *Let* f_i, $i = 1, 2$, *be in* $\mathcal{F}_0$; *then*

(1) $f_1 + f_2 \in \mathcal{F}_0$;

(2) *if* $f_1 \geqslant f_2$, *then* $f_1 - f_2 \in \mathcal{F}_0$;

(3) $f_1 \wedge f_2 \in \mathcal{F}_0$;

(4) $\alpha(f_i) \in \mathcal{F}_0$, $i = 1, 2$ (*where* $\alpha(\infty \cdot N_\infty) = \infty \cdot \alpha(N_\infty)$).

We now proceed to define, for each $f \in \mathcal{F}_0$, two other elements of $\mathcal{F}_0$, to be denoted by $\beta(f)$ and $\gamma(f)$, that will play an essential role in the proof of the main result.

But first we define, by induction, a sequence $\{f_i\}_{i=0}^\infty$ of functions in $\mathfrak{X}$ as follows:

$$f_0 = f \wedge 1, \qquad f_k = \left(f - \sum_{n=0}^{k-1} f_n \right) \wedge \left(1 - \sum_{n=0}^{k-1} \alpha^{k-n}(f_n) \right).$$

For each k, f_k lies in $\mathcal{F}_0$ (Lemma 3.3) and we can define $\beta(f)$ to be $\sum_{n=0}^\infty f_n = \sup_k \sum_{n=0}^k f_n$. For each $k \geqslant 0$, $f_k \leqslant f - \sum_{n=0}^{k-1} f_n$; hence $\beta(f) \leqslant f$. Similarly, for each $k \geqslant 0$, $f_k \leqslant 1 - \sum_{n=0}^{k-1} \alpha^{k-n}(f_n)$; hence,

$$\sum_{n=0}^{k} \alpha^{-n}(f_n) = \alpha^{-k}\left(f_k + \sum_{n=0}^{k-1} \alpha^{k-n}(f_n) \right) \leqslant \alpha^{-k}(1) = 1;$$

thus we can define $\gamma(f)$ to be $\sum_{n=0}^k \alpha^{-n}(f_n)$ $(= \sup_k \sum_{n=0}^k \alpha^{-n}(f_n))$ and $\gamma(f) \leqslant 1$.

LEMMA 3.4. *For each $f \in \mathfrak{F}_0$ there are projections $\{Q_n\}_{n=0}^{\infty}$ with the following properties*:

(1) $Q_n \in E_0 \mathfrak{L}(M)' E_0$, $Q_i Q_j = 0$ *if* $i \neq j$;

(2) $\phi(\sum_{n=0}^{\infty} Q_n) = L_{\gamma(f)}$;

(3) $\phi(\sum_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}) = L_{\beta(f)}$.

Therefore, $L_{\beta(f)}$ *and* $L_{\gamma(f)}$ *lie in* $\phi(\mathfrak{P}_1)$ (*recall that* $\mathfrak{P}_1$ *is the set of the wandering projections in* $\mathfrak{L}(M)'$).

PROOF. For each $n \geq 0$, $\alpha^{-n}(f_n)$ lies in $\mathfrak{F}_0$ (by Lemma 3.3). Also we have $\alpha^{-n}(f_n) \leq 1$; hence $\alpha^{-n}(f_n)$ lies in $\mathcal{C}$ and, by the definition of $\mathcal{C}$, there is a projection Q_n in $E_0 \mathfrak{L}(M)' E_0$ with $\phi(Q_n) = L_{\alpha^{-n}(f_n)}$. Since $\sum_{n=0}^{\infty} \alpha^{-n}(f_n) \leq 1$, we can choose $\{Q_n\}_{n=0}^{\infty}$ in such a manner that $Q_n Q_m = 0$ whenever $n \neq m$. From Lemma 2.4 we get

$$\phi(L_\delta^n Q_n L_\delta^{-n}) = L_{f_n}$$

and $\phi(\sum_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}) = L_{\beta(f)}$ (note that $\{L_\delta^n Q_n L_\delta^{-n}\}_{n=0}^{\infty}$ is an orthogonal family of projections since $L_\delta^n Q_k L_\delta^{-n} \leq E_n$). It is only left to prove that $\sum_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}$ lies in $\mathfrak{P}_1$ ($\sum_{n=0}^{\infty} Q_n$ clearly does, since $Q_n \leq E_0$ for each n). For this, it suffices to show that, for each $k \neq 0$, $L_\delta^k (L_\delta^n Q_n L_\delta^{-n}) L_\delta^{-k}$ is orthogonal to $L_\delta^m Q_m L_\delta^{-m}$ for every $n, m \in \mathbf{Z}$. But

$$L_\delta^k (L_\delta^n Q_n L_\delta^{-n}) L_\delta^{-k} L_\delta^m Q_m L_\delta^{-m} \leq E_{k+n} E_m;$$

thus we can take $k = m - n$, and then we have $L_\delta^m Q_n Q_m L_\delta^{-m} = 0$ (since $0 \neq k = n - m$).

Note that, since $\phi(\mathfrak{P}_1) \subseteq \phi(\mathfrak{P}_0) = \mathfrak{L}(\mathfrak{F}_0)$, $\beta(f)$ and $\gamma(f)$ lie in $\mathfrak{F}_0$. □

LEMMA 3.5. *For $f \in \mathfrak{F}_0$, the following statements hold*:

(1) *If, for each projection* $N \in C$ (*i.e.*, $N \in Z$ *and* $\alpha(N) = N$), $\varphi(fN) \leq \varphi(N)$, *then* $\beta(f) = f$.

(2) *For each projection* $N \in C$, $\varphi(\beta(f)N) = \varphi(\gamma(f)N)$.

(3) *If, for some* $N \in C$, $\varphi(fN) > \varphi(N)$, *then there is another projection* $Q \in C$, $Q \leq N$, *such that* $\gamma(f)Q = Q$ *and* $\beta(f)Q < fQ$.

(4) *Both* $\gamma(f)$ *and* $\beta(f)$ *lie in* $\mathfrak{F}_1$.

PROOF. Fix $f \in \mathfrak{F}_0$. By definition, $\beta(f) = \sum_{n=0}^{\infty} f_n$ and $\gamma(f) = \sum_{n=0}^{\infty} \alpha^{-n}(f_n)$ for the sequence $\{f_n\}_{n=0}^{\infty}$ constructed by induction from f. For $N \in C$, $\beta(f)N = \sum f_n N$ and $\gamma(f)N = \sum \alpha^{-n}(f_n)N$. Hence,

$$\varphi(\beta(f)N) = \sum \varphi(f_n N) = \sum \varphi(\alpha^{-n}(f_n N)) = \sum \varphi(\alpha^{-n}(f_n)N) = \varphi(\gamma(f)N).$$

This proves (2). To prove (1) assume $\varphi(fN) \leq \varphi(N)$ for each $N \in C$, and $\beta(f) \neq f$. Let $\hat{N}_0$ be the set $\{x \in X: \beta(f)(x) < f(x)\}$; then $\nu(\hat{N}_0) > 0$. In particular, $f > f_0$ on $\hat{N}_0$. But $f_0 = f \wedge 1$; hence $f_0 = 1$ on $\hat{N}_0$ and also $\gamma(f) = 1$ on $\hat{N}_0$. For $k > 0$, $\sum_{n=0}^{k} f_n \leq \beta(f) < f$ on $\hat{N}_0$ implies that $f_k < f - \sum_{n=0}^{k-1} f_n$ there. Therefore, on $\hat{N}_0$, $f_k = 1 - \sum_{n=0}^{k-1} \alpha^{k-n}(f_n)$ and, consequently, $\sum_{n=0}^{k} \alpha^{-n}(f) = 1$ and $\gamma(f) = 1$ on $\alpha^{-k}(\hat{N}_0)$. Let $\hat{N}$ be the set $\bigcup_{k=0}^{\infty} \alpha^{-k}(\hat{N}_0)$. Then $\gamma(f) = 1$ on $\hat{N}$ and N lies in C. Since $\beta(f) < f$ on $\hat{N}_0 \subseteq \hat{N}$, we have $\varphi(\beta(f)N) < \varphi(fN)$ ($\hat{N}_0$ was assumed to have a positive measure). Therefore,

$$\varphi(fN) > \varphi(\beta(f)N) = \varphi(\gamma(f)N) = \varphi(N).$$

The last equality follows from the fact that $\gamma(f)=1$ on $\hat{N}$. We obtained a contradiction and, thus, proved (1).

Now assume $\varphi(fN)>\varphi(N)$ for some projection $N\in C$. Let $\hat{N}_0$ be the set $\{x\in\hat{N}\colon \gamma(f)(x)<1\}$. In particular, $f_0<1$ on $\hat{N}_0$, but since $f_0=f\wedge 1$, $f_0=f$ and also $\beta(f)=f$ on $\hat{N}_0$. For $k>0$, $\Sigma_{n=0}^{k}\alpha^{-n}(f_n)<1$ on $\hat{N}_0$ implies $f_k<1-\Sigma_{n=0}^{k-1}\alpha^{k-n}(f_n)$ on $\alpha^k(\hat{N}_0)$ and, consequently, $\beta(f)=f$ on $\bigcup_{k=0}^{\infty}\alpha^k(\hat{N}_0)$ (to be denoted by $\hat{N}_1$). Since

$$\varphi(fN_1)=\varphi(\beta(f)N_1)=\varphi(\gamma(f)N_1)<\varphi(N_1)\quad\text{and}\quad\varphi(fN)>\varphi(N),$$

we have $\varphi(f(N-N_1))>\varphi(N-N_1)$ (note that $\hat{N}_1\subseteq\hat{N}$ and $N_1\in C$). Let Q denote the projection $N-N_1$. Then Q lies in C and $\gamma(f)=1$ on $\hat{Q}$ (since $\hat{N}_0\subseteq\hat{N}_1$). If $\beta(f)Q=fQ$, then $\varphi(fQ)=\varphi(\beta(f)Q)=\varphi(\gamma(f)Q)=\varphi(Q)$, contradicting what we have seen above. Therefore, $\beta(f)Q<fQ$.

We now turn to statement (4). Since $\gamma(f)<1$, clearly $\varphi(\gamma(f)N)\leqslant\varphi(N)$ for each $N\in C$. For $\beta(f)$ we have $\varphi(\beta(f)N)=\varphi(\gamma(f)N)\leqslant\varphi(N)$; hence $\beta(f)$ also lies in $\mathcal{F}_1$. $\square$

LEMMA 3.6. *Let Q be a projection in C and F_i, $i=1,2$, a projection in $\mathcal{P}_1$ satisfying the following conditions*:

(i) $\phi(F_1L_Q)\leqslant\phi(F_2L_Q)<\infty$, *and*

(ii) $\Sigma_{n=-\infty}^{\infty}L_\delta^nF_1L_QL_\delta^{-n}=L_Q$.

Then there is a unitary operator $U\in\mathfrak{R}$ such that $UF_1L_QU^*=F_2L_Q$ and $\phi(F_1L_Q)=\phi(F_2L_Q)$.

PROOF. Since $\phi(F_1L_Q)\leqslant\phi(F_2L_Q)<\infty$, there exist a projection E in $\mathfrak{L}(M)'$ and a partial isometry W in $\mathfrak{L}(M)'$ such that $F_1L_Q=W^*W$, $E=WW^*\leqslant F_2L_Q$. Let U_0 be the operator $\Sigma_{n=-\infty}^{\infty}L_\delta^nWL_\delta^{-n}$. Since W is a partial isometry with initial projection F_1L_Q and final projection E, $L_\delta^nWL_\delta^{-n}$ is a partial isometry (in $\mathfrak{L}(M)'$) with initial projection $L_\delta^nF_1L_QL_\delta^{-n}$ and final projection $L_\delta^nEL_\delta^{-n}$. Therefore, U_0 is a partial isometry in $\mathfrak{L}(M)'$ with initial projection L_Q (condition (ii)) and final projection

$$\sum_{n=-\infty}^{\infty}L_\delta^nEL_\delta^{-n}\leqslant\sum_{n=-\infty}^{\infty}L_\delta^nF_2L_QL_\delta^{-n}\leqslant L_Q.$$

Clearly, U_0 lies in $\mathfrak{R}$ and, thus, from finiteness of $\mathfrak{R}$, we get that its final projection is L_Q. Now let U be the unitary operator (in $\mathfrak{R}$) defined by $U=U_0+(I-L_Q)$, and we have

$$UF_1L_QU^*=U_0F_1L_QU_0^*=WF_1L_QW^*=E$$

and

$$\sum_{n=-\infty}^{\infty}L_\delta^nEL_\delta^{-n}=\sum L_\delta^nUF_1L_QU^*L_\delta^{-n}=U\Big(\sum L_\delta^nF_1L_QL_\delta^{-n}\Big)U^*=UL_QU^*=L_Q.$$

In order to show $\phi(F_2L_Q)=\phi(F_1L_Q)$, we will show $F_2L_Q=E$. For this, define $P=F_2L_Q-E$; then $PE=0$ and, for each $n\in Z$, $n\neq 0$, P is orthogonal to $L_\delta^nEL_\delta^{-n}$ because $P\leqslant F_2L_Q$, $L_\delta^nEL_\delta^{-n}\leqslant L_\delta^nF_2L_QL_\delta^{-n}$ and F_2L_Q is a wandering projection. Since $\Sigma_{n=-\infty}^{\infty}L_\delta^nEL_\delta^{-n}=L_Q$ and $P=PL_Q$, $P=0$, and we are done. $\square$

THEOREM 3.7. $\phi(\mathcal{P}_1) = \mathfrak{L}(\mathcal{F}_1)$.

PROOF. (a) Suppose f lies in $\mathcal{F}_1$; then, by Lemma 3.5(1), $f = \beta(f)$. By Lemma 3.4, $L_f = L_{\beta(f)}$ lies in $\phi(\mathcal{P}_1)$.

(b) Suppose now that P lies in $\mathcal{P}_1$ and $L_f = \phi(P)$. By Lemma 3.4 there is an orthogonal family of projections $\{Q_n\}_{n=0}^{\infty}$ in $E_0\mathfrak{L}(M)'E_0$ with $\phi(\Sigma Q_n) = L_{\gamma(f)}$ and $\phi(\Sigma L_\delta^n Q_n L_\delta^{-n}) = L_{\beta(f)}$. Denote ΣQ_n by E and $\Sigma L_\delta^n Q_n L_\delta^{-n}$ by F.

If $f \notin \mathcal{F}_1$, then there is some projection $Q \in C$ such that $\gamma(f)Q = Q$ and $\beta(f)Q < fQ$ (Lemma 3.5(3)). Consequently,

$$\phi(EL_Q) = L_{\gamma(f)}L_Q = L_Q = \phi(E_0L_Q)$$

and

$$\phi(FQ) = L_{\beta(f)}L_Q < L_fL_Q = \phi(PQ).$$

We also have $EL_Q \leqslant E_0L_Q$, but since E_0 is a finite projection (in $\mathfrak{L}(M)'$) and $\phi(EL_Q) = \phi(E_0L_Q)$ implies $EL_Q \sim E_0L_Q$, then $E_0L_Q = EL_Q$. Consequently,

$$\sum_{n=-\infty}^{\infty} L_\delta^n EL_Q L_\delta^{-n} = \sum_{n=-\infty}^{\infty} L_\delta^n E_0 L_Q L_\delta^{-n} = \sum_{n=-\infty}^{\infty} L_\delta^n E_0 L_\delta^{-n} L_Q = L_Q.$$

Also,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} L_\delta^n FL_Q L_\delta^{-n} &= \sum_{n=-\infty}^{\infty} L_\delta^n \left(\sum_{k=0}^{\infty} L_\delta^k Q_k L_\delta^{-k} \right) L_\delta^{-n} L_Q \\ &= \sum_{n=-\infty}^{\infty} L_\delta^n \left(\sum_{k=0}^{\infty} Q_k \right) L_\delta^{-n} L_Q = \sum L_\delta^n EL_Q L_\delta^{-n} = L_Q.\end{aligned}$$

Now we can use Lemma 3.6, with F, P in place of F_1, F_2, and we find that $\phi(FL_Q) = \phi(PL_Q)$, which is a contradiction. □

PROPOSITION 3.8. *Suppose f lies in $\mathcal{F}_1$ and, for every projection N, in C, $\varphi(fN) = \varphi(N)$. Then there is some $P \in \mathcal{P}_1$ with $\phi(P) = L_f$ and $\Sigma_{n=-\infty}^{\infty} L_\delta^n P L_\delta^{-n} = I$ (i.e., the corresponding subspace is full).*

PROOF. From Lemma 3.5(1) we know that $f = \beta(f)$. Therefore, $1 = \varphi(I) = \varphi(f) = \varphi(\beta(f)) = \varphi(\gamma(f))$. But $\gamma(f) \leqslant 1$; hence $\gamma(f) = 1$. Hence, with the notations of Lemma 3.4, $\Sigma_{n=0}^{\infty} Q_n = I$ and $\phi(\Sigma_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}) = L_f$. An easy calculation shows that

$$\sum_{k=-\infty}^{\infty} L_\delta^k \left(\sum_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n} \right) L^{-k} = I,$$

and this completes the proof ($P = \Sigma_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}$). □

Let $\mathcal{F}_2$ denote the set of all $f \in \mathcal{F}_1$ with $\varphi(fN) = \varphi(N)$ for each projection $N \in C$. Then we have

COROLLARY 3.9. $\mathfrak{L}(\mathcal{F}_2) = \phi(\{P \in \mathcal{P}_1\text{: } \mathfrak{M}(P) \textit{ is full}\})$.

PROOF. One inclusion was already proved in Proposition 3.8. For the other, let $P \in \mathcal{P}_1$ be a projection such that $\mathfrak{M}(P)$ is full, i.e., $\Sigma_{n=-\infty}^{\infty} L_\delta^n P L_\delta^{-n} = I$. Since

$P \in \mathcal{P}_1$, $\phi(P) = L_f$ and $f \in \mathcal{F}_1$. If for some projection $N \in C$, $\varphi(fN) < \varphi(N)$, then $\phi(PL_N) = L_f L_N < L_N = \phi(E_0 L_N)$. We can use Lemma 3.6 to obtain $\phi(PL_N) = \phi(E_0 L_N)$, which is a contradiction. □

The next proposition was proved in [**4**]. We obtain it here as a special case of our analysis.

PROPOSITION 3.10. (1) *α fixes Z elementwise if and only if every pure invariant subspace of L^2 has the form $R_v H^2$ for some partial isometry v in L^∞.* (2) *M is a factor if and only if $C = \{0, I\}$ and each pure invariant subspace of L^2 has the form $R_v H^2$ for some partial isometry v in L^∞.*

PROOF. (2) follows immediately from (1); hence it suffices to prove (1).

Assume α fixes Z elementwise; then $\mathcal{F}_1 = \{f \in \mathcal{F}_0 \colon f \leq 1\}$, and for every pure invariant subspace $\mathfrak{M}$ of L^2 we have

$$\phi(P(\mathfrak{M})) \leq 1 = \phi(E_0) = \phi(P(H^2)).$$

From Corollary 2.5 it follows that $\mathfrak{M} = R_v H^2$ for some partial isometry v in L^∞.

For the other direction, if α does not fix Z elementwise, then there is some nonzero projection N in Z with $\alpha(N)N = 0$. Let f, in Z, be $1 + N - \alpha(N)$; then f lies in $\mathcal{F}_2$ and, thus, $L_f = \phi(P)$ for some $P \in \mathcal{P}_1$ with a corresponding invariant subspace M that is full. If $\mathfrak{M} = R_v H^2$ for some partial isometry $v \in L^\infty$, then $L_\delta^n \mathfrak{M} = R_v L_\delta^n H^2$ for each $n \in Z$. As $\mathfrak{M}$ is full, $R_v L^2 = L^2$, and R_v is a unitary operator because $\mathfrak{R}$ is finite. Therefore, if we let Q be the projection onto $\mathfrak{M}$, and Q_0 is the one onto H^2, then $Q = R_v Q_0 R_v^*$ and, consequently (Corollary 2.5), $\phi(P) \leq \phi(E_0) = 1$. But $\phi(P) = L_f \not\leq 1$. Therefore, $\mathfrak{M}$ is not of the form $R_v H^2$. □

4. Subalgebras of $\mathfrak{L}$. In [**4**] it was shown that, when α fixes the center of M, every σ-weakly closed subalgebra of $\mathfrak{L}$ which contains $\mathfrak{L}_+$ is of the form $(1 - L_e)\mathfrak{L} \oplus L_e \mathfrak{L}_+$ for some projection e in C. We shall extend this result here.

Let $\mathfrak{M}$ be a pure invariant subspace of L^2; then $\mathfrak{B}(\mathfrak{M}) = \{L_f \in \mathfrak{L} \colon L_f \mathfrak{M} \subseteq \mathfrak{M}\}$ is a σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $\mathfrak{L}_+$. Conversely, let $\mathfrak{B}$ be such an algebra and $\mathfrak{M}_1$ the subspace $[\mathfrak{B}]_2$ (i.e., the closure, in L^2, of $\mathfrak{B}$). Then $\mathfrak{M}_1$ is an invariant subspace and $\mathfrak{M} = \mathfrak{M}_1 \ominus \bigcap_{n \geq 0} L_\delta^n \mathfrak{M}_1$ is a pure invariant subspace of L^2. By [**6**, Theorem 1], $\mathfrak{B} = \mathfrak{B}(\mathfrak{M}_1)$ (because $L_f[\mathfrak{B}]_2 \subseteq [\mathfrak{B}]_2$ implies $L_f \psi \in [\mathfrak{B}]_2$; hence $L_f \in \mathfrak{B}$). Also, the subspace $\bigcap_{n \geq 0} L_\delta^n \mathfrak{M}_1$ is a reducing subspace for $\mathfrak{L}_+$; hence it is the range of a projection in $\mathfrak{L}'$ $(= \mathfrak{R})$ and, consequently, $\mathfrak{B}(\mathfrak{M}) = \mathfrak{B}(\mathfrak{M}_1) = \mathfrak{B}$.

We conclude that, in order to obtain all the σ-weakly closed subalgebras of $\mathfrak{L}$ containing $\mathfrak{L}_+$, it is enough to consider those of the form $\mathfrak{B}(\mathfrak{M})$ for some pure invariant subspace $\mathfrak{M}$.

Suppose $\mathfrak{M}_i$ is a pure invariant subspace, for $i = 1, 2$, with $\phi(P(\mathfrak{M}_1)) = \phi(P(\mathfrak{M}_2))$. Let Q_i be the orthogonal projection onto $\mathfrak{M}_i$; then, by the discussion in §2, $Q_2 = R_v Q_1 R_v^*$ for some partial isometry $v \in L^\infty$, and $R_v^* R_v \geq Q_1$. In this case

$$\begin{aligned}\mathfrak{B}(\mathfrak{M}_2) &= \{T \in \mathfrak{L} \colon Q_2 T Q_2 = T Q_2\} = \{T \in \mathfrak{L} \colon R_v Q_1 R_v^* T R_v Q_1 R_v^* = T R_v Q_1 R_v^*\} \\ &= \{T \colon R_v Q_1 T Q_1 R_v^* = R_v T Q R_v^*\} \\ &\subseteq \{T \colon R_v^* R_v Q_1 T Q_1 R_v^* R_v = R_v^* R_v T Q_1 R_v^* R_v\} = \mathfrak{B}(\mathfrak{M}_1).\end{aligned}$$

Hence, by symmetry, $\mathfrak{B}(\mathfrak{M}_1) = \mathfrak{B}(\mathfrak{M}_2)$. It suffices, therefore, to consider, for each $f \in \mathfrak{F}_1$, just a single pure invariant subspace $\mathfrak{M}$ with $\phi(P(\mathfrak{M})) = L_f$. Such a subspace can be obtained by Theorem 3.7. The set of subspaces that can be obtained by the procedure of Theorem 3.7 will be denoted by $\mathbb{S}$.

LEMMA 4.1. *$W_t\mathfrak{M} = \mathfrak{M}$ for each $t \in \mathbf{R}$ and $\mathfrak{M} \in \mathbb{S}$. Consequently, $\beta_t(\mathfrak{B}(\mathfrak{M})) = \mathfrak{B}(\mathfrak{M})$ and $\varepsilon_n(\mathfrak{B}(\mathfrak{M})) \subseteq \mathfrak{B}(\mathfrak{M})$, $n \in \mathbf{Z}$.*

PROOF. Let P be $P(\mathfrak{M})$ for some $\mathfrak{M} \in \mathbb{S}$. Then $P = \sum_{n=0}^{\infty} L_\delta^n Q_n L_\delta^{-n}$ for an orthogonal family $\{Q_n\}_{n=0}^{\infty}$ of projections in $E_0\mathfrak{L}(M)'E_0$ (see Theorem 3.7). It is enough to show that $W_t L_\delta^n P L_\delta^{-n} W_t^* = L_\delta^n P L_\delta^{-n}$, $n \geqslant 0$, $t \in \mathbf{R}$, because $\mathfrak{M}$ is the range of $\sum_{n=0}^{\infty} L_\delta^n P L_\delta^{-n}$. We fix $n \geqslant 0$ and t in $\mathbf{R}$, and note that

$$W_t = \sum_{k=-\infty}^{\infty} e^{2\pi ikt}E_k \quad \text{and} \quad L_\delta^n P L_\delta^{-n} = \sum_{k=0} L_\delta^{n+k} Q_k L_\delta^{-n-k};$$

then

$$\begin{aligned} W_t L_\delta^n P L_\delta^{-n} W_t^* &= \sum_{\substack{m,k \in \mathbf{Z},\\ l \geqslant 0}} e^{2\pi ikt} E_k L_\delta^{n+l} Q L_\delta^{-n-l} e^{-2\pi imt} E_m \\ &= \sum_{l=0}^{\infty} e^{2\pi i(n+l)t} L_\delta^{n+l} Q_l L_\delta^{-n-l} e^{-2\pi i(n+l)t} = L_\delta^n P L_\delta^{-n}. \end{aligned}$$

The second equality above follows from the fact that $L_\delta^{n+l} Q_l L_\delta^{-n-l}$ is in $E_{n+l}\mathfrak{L}(M)'E_{n+l}$. Therefore, $W_t\mathfrak{M} = \mathfrak{M}$ for $t \in \mathbf{R}$ and $\mathfrak{M} \in \mathbb{S}$. Since $\beta_t(L_f) = W_t L_f W_t^*$, it follows that $\beta_t(\mathfrak{B}(\mathfrak{M})) = \mathfrak{B}(\mathfrak{M})$, and the last assertion follows from the representation of ε_n as the integral (convergent in the σ-weak topology when applied to operators) $\int_0^1 e^{-2\pi int}\beta_t\,dt$. $\square$

LEMMA 4.2. *For $x \in M$ and $k > 0$, $L_x L_\delta^{-k}$ lies in $\mathfrak{B}(\mathfrak{M})$ (where $\mathfrak{M}$ is a pure invariant subspace in $\mathbb{S}$) if and only if $x \in \bigcap_{m=1}^{k} \alpha^{-m}(A)$ where $A = \{x \in M: L_x P = 0\}$ ($P = P(\mathfrak{M})$).*

PROOF. As was seen before, $P = \sum_{l=0}^{\infty} L_\delta^l Q_l L_\delta^{-l}$ for some orthogonal family of projections $\{Q_n\}_{n=0}^{\infty}$ in $E_0\mathfrak{L}(M)'E_0$, and $Q = \sum_{n=0}^{\infty} L_\delta^n P L_\delta^{-n}$ is the projection onto $\mathfrak{M}$. Then

$$Q = \sum_{n=0}^{\infty} L_\delta^n P L_\delta^{-n} = \sum_{n=0}^{\infty} L_\delta^n \left(\sum_{l=0}^{\infty} L_\delta^l Q_l L_\delta^{-l} \right) L_\delta^{-n} = \sum_{l=0}^{\infty} L_\delta^l C_l L_\delta^{-l},$$

where $C_l = \sum_{n=0}^{l} Q_n$. Let $\mathfrak{M}_n$ be the range of $L_\delta^n C_n L_\delta^{-n}$; then $\mathfrak{M} = \sum_{n=0}^{\infty} \oplus \mathfrak{M}_n$, $\mathfrak{M}_n \subseteq E_n(L^2)$, and $L_x L_\delta^{-k}\mathfrak{M}_n \subseteq E_{n-k}(L^2)$. Hence, $L_x L_\delta^{-k}$ maps $\mathfrak{M}$ into itself if and only if it maps $\mathfrak{M}_n$ into $\mathfrak{M}_{n-k}$ for each $n \geqslant 0$ (where we let $\mathfrak{M}_l = \{0\}$ if $l < 0$). Let $\mathfrak{N}_m$ be the range of Q_m, for $m \geqslant 0$; then $\mathfrak{M}_n = \sum_{m=0}^{n} \oplus L_\delta^n \mathfrak{N}_m$. For a fixed $n \geqslant k$, $L_x L_\delta^{-k}$ maps $\mathfrak{M}_n$ into $\mathfrak{M}_{n-k}$ if and only if

$$L_x L_\delta^{-k} \sum_{m=0}^{n} \oplus L_\delta^n \mathfrak{N}_m \subseteq \sum_{m=0}^{n-k} \oplus L_\delta^{n-k} \mathfrak{N}_n.$$

This occurs if and only if $L_xL_\delta^{-k}$ vanishes on $L_\delta^n\mathfrak{N}_m$ for each $n \geq m > n-k$. For $k > n \geq 0$, $L_xL_\delta^{-k}$ maps $\mathfrak{M}_n$ into $\mathfrak{M}_{n-k}$ $(=\{0\})$ if and only if it vanishes on $L_\delta^n\mathfrak{N}_m$ for each $n \geq m \geq 0$.

Considering an arbitrary $n \geq 0$, we see that $L_xL_\delta^{-k}$ lies in $\mathfrak{B}(\mathfrak{M})$ if and only if it vanishes on $L_\delta^n\mathfrak{N}_m$ for each $n \geq 0$ and $n \geq m > \max(-1, n-k)$. We can write this condition in a different form. Let $\mathfrak{M}^1$ be the subspace $\sum_{n=0}^{\infty} \oplus L_\delta^n\mathfrak{N}_n$; then one can easily check that $L_xL_\delta^{-k}$ lies in $\mathfrak{B}(\mathfrak{M})$ if and only if it vanishes on $\sum_{m=0}^{k-1} L_\delta^m\mathfrak{M}^1$; i.e., if and only if L_x vanishes on $\sum_{m=1}^{k} L_\delta^{-m}\mathfrak{M}^1$. But $\mathfrak{M}^1$ is the range of P; hence $L_xL_\delta^{-k}$ lies in $\mathfrak{B}(\mathfrak{M})$ if and only if $L_xL_\delta^{-m}PL_\delta^m = 0$ for each $k \geq m \geq 1$; i.e., if and only if $L_{\alpha^m(x)}P = L_\delta^m(L_xL_\delta^{-m}PL_\delta^m)L_\delta^{-m} = 0$ for each $k \geq m \geq 1$. □

LEMMA 4.3. *For $P \in \mathfrak{P}_1$, denote by e the range projection of $\phi(P)$ ($e \in \mathfrak{L}(Z)$). Then* $\{x \in M: L_xP = 0\} = \{x \in M: L_x \in \mathfrak{L}(M)(1-e)\}$.

PROOF. Set $J = \{L_x \in \mathfrak{L}(M): L_xP = 0\}$; then J is a strongly closed two-sided ideal in $\mathfrak{L}(M)$ (since $P \in \mathfrak{L}(M)'$). Hence $J = \mathfrak{L}(M)N_0$ for some projection $N_0 \in \mathfrak{L}(Z)$. For each projection N, in $\mathfrak{L}(Z)$, N lies in J if and only if $NP = 0$. Since ϕ is faithful, this occurs if and only if $\phi(NP) = 0$; i.e., if and only if $N\phi(P) = 0$. Since N_0 is the largest projection in $\mathfrak{L}(Z) \cap J$, $N_0 = 1 - e$. □

Let e be a projection in $\mathfrak{L}(Z)$. We define $\mathfrak{B}(e)$ to be the following subset of $\mathfrak{L}$: $\mathfrak{B}(e) = \{T \in \mathfrak{L}: \varepsilon_k(T) \in (1-e_k)\mathfrak{L}(M)L_\delta^k$ for each $k \in Z\}$, where $e_k = \bigvee_{m=1}^{-k} \alpha^{-m}(e)$, for $k < 0$, and $e_k = 0$ for $k \geq 0$.

LEMMA 4.4. *The projections $\{e_k\}_{k=-\infty}^{\infty}$ defined above satisfy, for each $n, k \in \mathbf{Z}$,*

$$(*) \qquad e_n(1-e_k)\big(1-\alpha^k(e_{n-k})\big) = 0.$$

PROOF. We will distinguish between four cases:

(1) When $n \geq 0$, $e_n = 0$ and $(*)$ is obvious.

(2) When $n < 0$ and $k \geq 0$; then

$$e_k = 0, \quad e_n = \bigvee_{m=1}^{-n} \alpha^{-m}(e), \quad e_{n-k} = \bigvee_{m=1}^{k-n} \alpha^{-m}(e)$$

and

$$\alpha^k(e_{n-k}) = \bigvee_{m=1-k}^{-n} \alpha^{-m}(e) \geq \bigvee_{m=1}^{-n} \alpha^{-m}(e) = e_n.$$

Hence $e_n(1-\alpha^k(e_{n-k})) = 0$.

(3) When $k \leq n < 0$, then $e_{n-k} = 0$ and we have to show $e_n \leq e_k$. This is clear since $-n < -k$.

(4) When $n < k < 0$, then $e_n = \bigvee_{m=1}^{-n} \alpha^{-m}(e)$, $e_k = \bigvee_{m=1}^{-k} \alpha^{-m}(e)$ and $\alpha^k(e_{n-k}) = \bigvee_{m=1-k}^{-n} \alpha^{-m}(e)$. For $1 \leq m \leq -k$, $\alpha^{-m}(e) \leq e_k$ and for $-k+1 \leq m \leq -n$, $\alpha^{-m}(e) \leq \alpha^k(e_{n-k})$. In both cases $\alpha^{-m}(e)(1-e_k)(1-\alpha^k(e_{n-k})) = 0$. Now $(*)$ follows from the definition of e_n. □

REMARK. If f lies in L^∞, then $\varepsilon_k(L_f) = L_{f(k)}L_\delta^k$ for each $k \in \mathbf{Z}$. Indeed, for each $x \in M$ and each $k \in \mathbf{Z}$,

$$\Big(x\psi^*\underbrace{\delta^*\delta^*\cdots{}^*\delta}_{k\text{ times}}\Big)(n) = \begin{cases} x, & n = k, \\ 0, & n \neq k. \end{cases}$$

Hence,

$$\varepsilon_k(L_f) = L_{\varepsilon_{k(f)}} = L_{E_{k(f)}} = L_{f(k)\psi * \delta * \cdots * \delta} = L_{f(k)} L_\delta^k.$$

THEOREM 4.5. (1) *For each projection* e, *in* $\mathfrak{L}(Z)$, $\mathfrak{B}(e)$, *as defined above, is a* σ-*weakly closed subalgebra of* $\mathfrak{L}$ *that contains* $\mathfrak{L}_+$. (2) *For each* σ-*weakly closed subalgebra* $\mathfrak{B}$ *of* $\mathfrak{L}$, *that contains* $\mathfrak{L}_+$, *there is some projection* e, *in* $\mathfrak{L}(Z)$, *such that* $\mathfrak{B} = \mathfrak{B}(e)$.

PROOF. (1) Fix a projection e in $\mathfrak{L}(Z)$. $\mathfrak{B}(e)$ is clearly a linear subspace of $\mathfrak{L}$ and contains $\mathfrak{L}_+$ since $e_k = 0$ for $k \geq 0$. Now, let T and S, in $\mathfrak{L}$, be L_f and L_g, respectively, and assume they both lie in $\mathfrak{B}(e)$; i.e., for each $k \in \mathbf{Z}$, $\varepsilon_k(T) = L_{f(k)} L_\delta^k$ and $\varepsilon_k(S) = L_{g(k)} L_\delta^k$ both lie in $(1 - e_k)\mathfrak{L}(M) L_\delta^k$ (see the remark following Lemma 4.4). We wish to show that $\varepsilon_n(TS)$ lies in $(1 - e_n)\mathfrak{L}(M) L_\delta^n$ for $n \in Z$; this will prove that $\mathfrak{B}(e)$ is an algebra. But $\varepsilon_n(TS) = L_{(f*g)(n)} L_\delta^n$, where $(f*g)(n) = \sum_{n=-\infty}^{\infty} f(k)\alpha^k(g(n-k))$. Hence, it will suffice to show

$$L_{f(k)\alpha^k(g(n-k))} \in (1 - e_n)\mathfrak{L}(M).$$

Since $L_{f(k)} \in (1 - e_k)\mathfrak{L}(M)$ and $L_{g(n-k)} \in (1 - e_{n-k})\mathfrak{L}(M)$, this follows immediately from Lemma 4.4.

The algebra $\mathfrak{B}(e)$ is also σ-weakly closed since ε_k is σ-weakly continuous for each $k \in \mathbf{Z}$ and $(1 - e_k)\mathfrak{L}(M) L_\delta^k$ is a σ-weakly closed subspace.

(2) Let $\mathfrak{B}$ be a σ-weakly closed subalgebra of $\mathfrak{L}$ containing $\mathfrak{L}_+$. By the discussion preceding Lemma 4.1, there is a pure invariant subspace $\mathfrak{M} \in \mathfrak{S}$ with $\mathfrak{B} = \mathfrak{B}(\mathfrak{M})$. For $P = P(\mathfrak{M})$, let $e \in \mathfrak{L}(Z)$ be the range projection of $\phi(P)$. We claim that $\mathfrak{B} = \mathfrak{B}(e)$.

Let T be in $\mathfrak{B}$, then $\varepsilon_k(T) \in \mathfrak{B}$ for each $k \in \mathbf{Z}$ (Lemma 4.1); hence, $\varepsilon_k(T) = L_x L_\delta^k$, where x lies in $\bigcap_{m=1}^{-k} \alpha^{-m}(A)$, $A = \{x \in M\colon L_x P = 0\}$, when we understand the intersection over an empty index set as M (Lemma 4.2). Using Lemma 4.3 we find that $\varepsilon_k(T)$ lies in $\bigwedge_{m=1}^{-k}(1 - \alpha^{-m}(e))\mathfrak{L}(M) L_\delta^k$ (where the intersection is $\mathfrak{L}(M) L_\delta^k$ when $k \geq 0$). Thus $\varepsilon_k(T)$ lies in $(1 - e_k)\mathfrak{L}(M) L_\delta^k$, as defined following Lemma 4.3, and T lies in $\mathfrak{B}(e)$. Therefore $\mathfrak{B} \subseteq \mathfrak{B}(e)$.

Using Lemmas 4.3 and 4.2, again, we see that $\varepsilon_k(\mathfrak{B}(e)) \subseteq \mathfrak{B}$. Consider the subspaces $[\mathfrak{B}]_2$ and $[\mathfrak{B}(e)]_2$. We know $[\mathfrak{B}]_2 \subseteq [\mathfrak{B}(e)]_2$, since $\mathfrak{B} \subseteq \mathfrak{B}(e)$, and $E_k[\mathfrak{B}(e)] \subseteq [\mathfrak{B}]_2$. As $\Sigma E_k = I$, $[\mathfrak{B}(e)]_2 = [\mathfrak{B}]_2$. By [**6**, Theorem 1], $\mathfrak{B} = \mathfrak{B}(e)$. □

As a special case, we have (see [**4**, Theorem 3.3 and Corollary 3.5])

COROLLARY 4.6. (1) *Every* σ-*weakly closed subalgebra* $\mathfrak{B}$ *of* $\mathfrak{L}$ *which contains* $\mathfrak{L}_+$ *has the form* $\mathfrak{B} = (1 - L_e)\mathfrak{L} \oplus L_e\mathfrak{L}_+$ (*for some projection* $e \in C$) *if and only if* α *fixes* Z *elementwise*.

(2) $\mathfrak{L}_+$ *is a maximal* σ-*weakly closed subalgebra of* $\mathfrak{L}$ *if and only if* M *is a factor*.

PROOF. (1) Assume first that α fixes Z elementwise; then $\alpha^{-m}(e) = e$ for each $e \in Z$ and $e_k = e$ (with the notations of the preceding theorem). Hence, each such algebra is of the required form.

On the other hand, if α does not fix Z, then $\alpha(e)e = 0$ for some projection $e \in Z$ and $\mathfrak{B}(e)$ will not have the form $(1 - L_e)\mathfrak{L} + L_e\mathfrak{L}_+$.

(2) From Theorem 4.5, $\mathfrak{L}_+$ would be a maximal σ-weakly closed subalgebra if and only if $Z = \mathbf{C}I$. □

One can also check the following:

COROLLARY 4.7. *α acts ergodically on Z if and only if there is no σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $L_e\mathfrak{L} \oplus (1 - L_e)\mathfrak{L}_+$ (for some nonzero projection e in Z) and is different from $\mathfrak{L}$.*

We conclude with the following result.

PROPOSITION 4.8. *Let $\mathfrak{B}$ be a σ-weakly closed subalgebra of $\mathfrak{L}$ that contains $\mathfrak{L}_+$ and let $\mathfrak{M}$ be an invariant subspace. Then $\mathfrak{M}$ is $\mathfrak{B}$-invariant (i.e., $T\mathfrak{M} \subseteq \mathfrak{M}$ for each $T \in \mathfrak{B}$) if and only if $\mathfrak{M} \ominus L_\delta\mathfrak{M} \subseteq e(L^2)$, where $\mathfrak{B} = \mathfrak{B}(e)$.*

PROOF. We can write $\mathfrak{M}$ as an orthogonal sum of a pure invariant subspace and a reducing subspace. The latter is clearly $\mathfrak{B}$-invariant; hence we assume $\mathfrak{M}$ is pure. Then $\mathfrak{M}$ is $\mathfrak{B}$-invariant if and only if $\mathfrak{B}(\mathfrak{M}) \supseteq \mathfrak{B}$; i.e., if and only if $e_0 \leq e$, where e_0 is the range projection of $\phi(P(\mathfrak{M}))$. Since $\phi(e) = e \cdot \infty$, this is equivalent to $\phi(P(\mathfrak{M})) \leq \phi(e)$. As $\phi(P(\mathfrak{M})) < \infty$, this holds if and only if $P(\mathfrak{M}) \precsim e$ (in $\mathfrak{L}(M)'$). But e lies in the center of $\mathfrak{L}(M)'$ and thus $\mathfrak{M}$ is $\mathfrak{B}$-invariant if and only if $P(\mathfrak{M}) \leq e$. □

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